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A fixed domain method for diffusion with a moving boundary

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SUMMARY

The one-dimensional diffusion equation for a region with one fixed boundary and one unknown moving boundary is transformed to a non-linear equation on a fixed region by using the moving boundary position as the time variable. The boundary velocity becomes a second dependent variable, with dependence only on the new time variable. An implicit finite difference scheme, marching in time, is applied to a problem with known analytic solution to demonstrate the computing speed and accuracy of this approach, and also to a problem solved previously by variable time step methods. This transformation reduces any parabolic or elliptic system of equations on a domain with moving boundary, or with unknown free surface in two space variables, to a non-linear fixed domain system which has advantages for computation.

1. Introduction

A variety of diffusion problems with moving boundaries, and associated analytic and numerical methods, are described in the Oxford Conference Proceedings edited by Ockendon and Hodgkins [1], subsequently designated OH. Analytic solutions are possible for very few problems, and the general approach is necessarily numerical. Tayler [2] describes the general multi-phase Stefan problem, with several unknown moving interfaces, and typical interface conditions. A two-phase problem involves one moving interface, and a classical simplification treats only one phase with one fixed and one moving boundary. Crank [3] discusses finite difference methods for the classical one-dimensional diffusion problem, which include variable space steps and variable time steps, and a change of space variable to fix the moving boundary. The latter transformation was used in an application by Ferris [4], but leads to a non-linear equation involving both the moving boundary position and velocity explicitly, and in turn a non-trivial iteration at each time step to update the boundary position. It was incorporated also in a different treatment by Baumeister and Hoffman [5].

The question of 'best method' is discussed by Fox [6], who reviews the above methods, the interchange of dependent and independent variables, integral and integro-differential equation methods and variational methods. He notes that some methods are special to the problem (or class of problems) treated, and not easily extended, in particular, to more than one space variable. However, the classical one-dimensional diffusion problem reveals the difficulties, and inadequacies, in dealing with a moving boundary, and provides a useful first test for the various methods. The recent review by Furzeland [7] compares the merits of four different methods,

including the above fixed domain transformation, by applying various subsets to three one-phase and one two-phase Stefan problems. There is not a best method for all the problems. A more restricted comparison of different variable time step methods has been presented by Gupta and Kumar [8], essentially different iteration procedures to determine the required time step, who show that their version remains stable for longer times and gives good agreement with a solution obtained by an approximate integral method.

I will now present an alternative fixed domain transformation by replacing time by boundary position, appropriate to one moving boundary in one space dimension. There is no direct extension to more than one space variable, or to more than one moving boundary, but within these limitations it is not restricted to linear or single equations. The original formulation was for a steady state elliptic free surface problem in two space variables but is currently an approach to the coupled thermomechanical equations for a steady state ice-sheet. Here the lead order balances in a perturbation method based on the small surface slope, developed by Morland and Johnson [9, 10] and Johnson [11] for plane and axisymmetric flows under isothermal conditions, generate a parabolic system. The horizontal coordinate is equivalent to time in the diffusion problem, and the surface slope is equivalent to the boundary velocity.

This method introduces the boundary velocity, expressed as a function of boundary position, as an additional dependent variable, and the transformed equation is necessarily non-linear. The extra boundary condition, now on a fixed boundary, determines the boundary velocity, and it is demonstrated by comparison with an exact solution how a simple implicit finite difference scheme achieves accurate results. In the examples presented at most one iteration to update the extrapolated boundary velocity is required, and only during an initial time period. There is no record of the iteration count for the previous space variable transformation method, but there both boundary position and boundary velocity must be extrapolated and updated. Also the number of iterations required for the variable time step method Gupta and Kumar [8], to achieve prescribed accuracy is not recorded, but their solution is much closer to that given by the present method, over their limited time period, than to that given by the approximate integral method used for comparison.

An essential feature of the transformation is the monotonicity of boundary position with respect to time (or surface slope in a two space dimension problem) over the interval considered. This is common in free surface problems and one-phase diffusion, but Furzeland [7] warns that boundary motion can change direction frequently in two-phase problems, when the solution must be computed sequentially in the corresponding time intervals. Change-over conditions at zero velocity may give rise to numerical difficulties.

2. Moving boundary problems

The main features of a one-dimensional, one-phase, moving boundary diffusion problem are incorporated in the following equations for a variable $u(x, t)$ and unknown boundary $x = s(t)$:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < s(t), \qquad t > 0, \tag{1}$$

$$\frac{\partial u}{\partial x} = -f(t), \qquad x = 0, \qquad t>0, \tag{2}$$

$$u = p(x), \qquad \frac{\partial u}{\partial x} = -\frac{ds}{dt}, \qquad x = s(t), \qquad t>0, \tag{3}$$

$$u = g(x), \qquad t = 0, \qquad 0<x<s(0). \tag{4}$$

Gupta and Kumar [8] treat the same problem as Douglas and Gallie [12], namely,

$$f(t) = 1, \qquad p(x) = 0, \qquad s(0) = 0, \tag{5}$$

where the initial distribution $g(x)$ is absent. The position dependent condition $(3)_1$ could be replaced by a time dependent condition $\bar{p}(t)$ with little change in the subsequent analysis, and an alternative to condition $(3)_2$, independent of boundary velocity,

$$\frac{\partial u}{\partial x} = q(x) \text{ or } \bar{q}(t), \quad x = s(t), \tag{6}$$

will also be analysed.

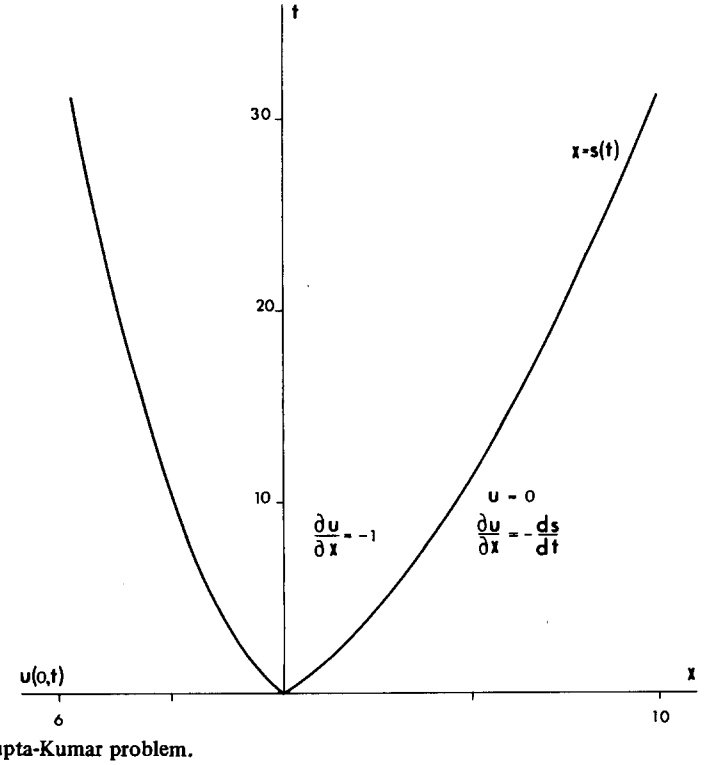

Figure 1. The Gupta-Kumar problem.

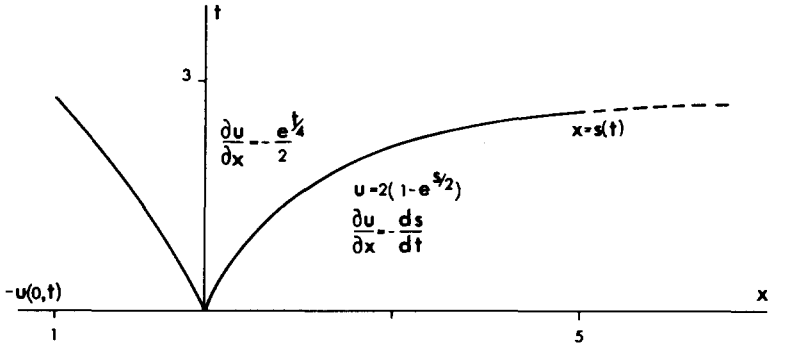

Figure 2. Boundary conditions with an exact solution.

Figure 1 illustrates the boundary conditions (5) of the GK problem, with the moving boundary and fixed boundary value $u(0, t)$ calculated by the present method. In this example the boundary velocity $\dot{s}$ decreases smoothly with t, from unity to 0.2135 at $t = 31.2$ as shown in later tables. Figure 2 illustrates the boundary conditions, moving boundary, and fixed boundary value $u(0, t)$ of an exact solution

$$u(x,t) = 1 - \exp(\tfrac{1}{2}x + \tfrac{1}{4}t), \qquad 0 \leqslant t < 4\ln 2, \tag{7}$$

$$s = -2\ln[2 - \exp(\tfrac{1}{4}t)], \qquad t = 4\ln[2 - \exp(-\tfrac{1}{2}s)], \tag{8}$$

$$\dot{s} = \exp(\tfrac{1}{2}s) - 0.5, \tag{9}$$

for

$$f(t) = \tfrac{1}{2}\exp(\tfrac{1}{4}t), \qquad p(s) = 2[1 - \exp(\tfrac{1}{2}s)], \qquad s(0) = 0. \tag{10}$$

As $t \to 4\ln 2 = 2.77$, $s \to \infty$ and $\dot{s} \to \infty$, and at the calculation limit $s = 10$, $t = 2.76$, $\dot{s} = 147.9$, with $\dot{s}(0) = 0.5$, so that a wide variation of boundary velocity is encompassed. Both examples have $s(0) = 0$, but initial conditions (4) for $s(0) > 0$ simply change the starting procedure for the finite difference scheme.

Now assume that there is a finite time over which the moving boundary does not change direction, so $s(t)$ is monotonic. Then s can replace t as an independent variable. In the above examples $\dot{s}(t) > 0$ for all boundary positions, so defining

$$u(x,t) = \hat{u}(x,s), \qquad \dot{s}(t) = \gamma(s) \geqslant 0, \tag{11}$$

the diffusion equation (1) becomes

$$\gamma \frac{\partial \hat{u}}{\partial s} = \frac{\partial^2 \hat{u}}{\partial x^2}, \qquad 0 < x < s. \tag{12}$$

More generally, $\dot{s}$ may change sign at $s = s_m$ and remain negative for another finite time range,

or be negative for some initial time range. Then equations (11) and (12) can be obtained in an increasing variable $\tilde{s} = -s$ with γ replaced by $\tilde{\gamma} = \dot{\tilde{s}}(t) \geqslant 0$, allowing forward integration in $\tilde{s}$. It is expected that $s(t)$ is monotonic for all, or long, times for a variety of boundary conditions.

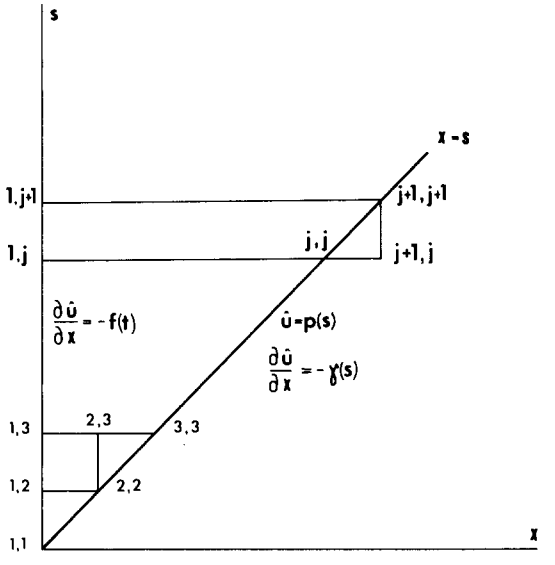

Figure 3. The fixed domain.

Figure 3 illustrates the fixed (x, s) domain and transformed boundary conditions (2) and (3), where the time t at each s is calculated by

$$t = \int_0^s \frac{ds'}{\gamma(s')} . \tag{13}$$

While the domain is now fixed, the boundary velocity $\gamma(s)$ is a new unknown dependent variable to be determined simultaneously with $\hat{u}(x, s)$, and the differential equation (12) is nonlinear.

3. Finite difference scheme

Equal x and s grid steps are the most convenient for the domain $0 < x < s$ with grid points (j, j) lying on the boundary $x = s$ (Figure 3). Let the step length be h and introduce the notation

$$\begin{aligned} x_i &= (i-1)h, \qquad s_j = (j-1)h, \\ u_{i,j} &= \hat{u}(x_i, s_j), \qquad \gamma_j = \gamma(s_j). \end{aligned} \qquad (i, j = 1, 2, \ldots) \tag{14}$$

Following Ames [13], an implicit finite difference approximation for the equation (12), with arbitrary weighting between the $j+1$ and j levels of s ($0 \leqslant r \leqslant 1, 0 \leqslant k \leqslant 1$), is

$$ru_{i-1,j+1} - (2r + \Gamma_j)u_{i,j+1} + ru_{i+1,j+1}$$

$$= -(1-r)u_{i-1,j} + [2(1-r) - \Gamma_j]u_{i,j} - (1-r)u_{i+1,j},$$

$$(i = 2, 3, \ldots j; j = 3, 4, \ldots), \tag{15}$$

where

$$\Gamma_j = kh\gamma_{j+1} + (1-k)h\gamma_j. \tag{16}$$

An explicit scheme for the values at s_{j+1} ($r = 0, k = 0$) is not generally satisfactory.

The boundary condition $(3)_1$ determines $u_{j+1,j+1} = p(s_{j+1})$ directly, and boundary condition $(2)_1$ has the difference approximation

$$u_{1,j+1} - u_{2,j+1} = hf(t_{j+1}). \tag{17}$$

Thus, if all values at s_j are known, including $u_{j+1,j}$ outside the domain, and γ_{j+1} and t_{j+1} are known, then (16), (15) are a conventional tridiagonal linear system of j equations for the j unknowns $u_{i,j+1}$ ($i = 1, 2, \ldots j$), which can be solved by a simple algorithm. The exterior value $u_{j+1,j}$ can be estimated by Taylor series extrapolation of the interior solution to $O(h^2)$ assuming that

$$p'(s) = b(s) \tag{18}$$

is order unity. From the boundary conditions (3) and differential equation (12),

$$\hat{u} = p, \quad \frac{\partial \hat{u}}{\partial x} = -\gamma, \quad \frac{\partial^2 \hat{u}}{\partial x^2} = \gamma^2 + \gamma b, \quad \text{on} \quad x = s, \tag{19}$$

and, hence,

$$u_{j+1,j} = p_j - h\gamma_j + \tfrac{1}{2}h^2(\gamma_j^2 + \gamma_j b_j). \tag{20}$$

Similarly, an inner expansion at level s_{j+1} gives a quadratic equation for γ_{j+1} with positive root

$$h\gamma_{j+1} = \{(1 + \tfrac{1}{2}hb_{j+1})^2 + 2(u_{j,j+1} - p_{j+1})\}^{1/2} - (1 + \tfrac{1}{2}hb_{j+1}), \tag{21}$$

which is used to update γ_{j+1} once $u_{j,j+1}$ is determined from the above linear system. An iteration is started with $\Gamma_j = h\gamma_j$. The time t at each s, and in particular t_{j+1}, required in the boundary condition (17), is calculated by

$$t_1 = 0, \quad t_2 = \frac{h}{3}\left(\frac{1}{\gamma_2} + \frac{4}{\gamma_{3/2}} + \frac{1}{\gamma_1}\right), \tag{22}$$

$$t_{j+1} - t_{j-1} = \frac{h}{3}\left(\frac{1}{\gamma_{j+1}} + \frac{4}{\gamma_j} + \frac{1}{\gamma_{j-1}}\right), \qquad (j = 2, 3, \ldots), \tag{22}$$

where the updated γ_{j+1} is used at each step of the iteration. The iteration is halted when the Nth value $\gamma_{j+1}^{(N)}$ satisfies

$$\left|\gamma_{j+1}^{(N)} - \gamma_{j+1}^{(N-1)}\right| < h^2 \left|\gamma_{j+1}^{(N-1)}\right|. \tag{23}$$

For boundary conditions $(3)_1$, $(6)_1$, the expressions (20), (21) are replaced by

$$\begin{aligned} &u_{j+1,j} = p_j + hq_j + \tfrac{1}{2}h^2 \gamma_j (b_j - q_j), \\ &\tfrac{1}{2}h^2 (b_{j+1} - q_{j+1}) \gamma_{j+1} = u_{j,j+1} - p_{j+1} + hq_{j+1}, \end{aligned} \tag{24}$$

when $b_{j+1} - q_{j+1} = (\partial \hat{u}/\partial s)_{x = s_{j+1}} \neq 0$. For time dependent conditions $\bar{p}(t)$, $\bar{q}(t)$, with $\bar{p}'(t) = \bar{b}(t)$, the expressions are

$$\begin{aligned} &u_{j+1,j} = \bar{p}_j + hq_j + \tfrac{1}{2}h^2 (\bar{b}_j - \gamma_j \bar{q}_j), \\ &\tfrac{1}{2}h^2 \bar{q}_{j+1} \gamma_{j+1} = \tfrac{1}{2}h^2 \bar{b}_{j+1} - h\bar{q}_{j+1} + \bar{p}_{j+1} - u_{j,j+1}, \end{aligned} \tag{25}$$

when $\bar{q}_{j+1} = (\partial u/\partial x)_{x = s_{j+1}} \neq 0$. Here $\bar{p}_{j+1}$, $\bar{q}_{j+1}$, $\bar{b}_{j+1}$ are values at t_{j+1} to be updated at each iteration.

It remains to obtain expressions for $u_{1,2}$, and for $u_{1,3}$, $u_{2,3}$, γ_2, γ_3, required to start the s-marching with $j = 3$ in (15), (16), (17), (21), and for γ_1 to determine t_2, t_3. For smooth boundary conditions (2), (3), Taylor series expansions in (x, s) about $(0, 0)$ to $0(h^3)$ for $\hat{u}$ and to $O(h^2)$ for γ provide $O(h^3)$ accuracy in the starting values for (15), and t_2, t_3, which is better than the finite difference approximation. Values of the required derivatives at (0, 0) are obtained from continued differentiation of boundary conditions and differential equations. For the Gupta and Kumar [8] problem (5),

$$\begin{aligned} &\gamma_1 = 1, \quad \gamma_{3/2} = 1 - \tfrac{1}{2}h + \tfrac{1}{2}h^2, \quad \gamma_2 = 1 - h + 2h^2, \quad \gamma_3 = 1 - 2h + 8h^2, \\ &t_2 = h + \tfrac{1}{2}h^2 - \tfrac{1}{3}h^3, \quad t_3 = 2h + 2h^2 - \tfrac{8}{3}h^3, \\ &u_{1,2} = h - \tfrac{1}{2}h^2 + h^3, \quad u_{1,3} = 2h - 2h^2 + 8h^3, \quad u_{2,3} = h - \tfrac{3}{2}h^2 + 6h^3. \end{aligned} \tag{26}$$

For the boundary conditions (10),

$$\begin{aligned} &\gamma_1 = \tfrac{1}{2}, \quad \gamma_{3/2} = \tfrac{1}{2} + \tfrac{1}{4}h + \tfrac{1}{32}h^2, \quad \gamma_2 = \tfrac{1}{2} + \tfrac{1}{2}h + \tfrac{1}{8}h^2, \quad \gamma_3 = \tfrac{1}{2} + h + \tfrac{1}{2}h^2, \\ &t_2 = 2h - h^2 + \tfrac{1}{2}h^3, \quad t_3 = 4h - 4h^2 + 4h^3, \end{aligned} \tag{27}$$

$$u_{1,2} = -\tfrac{1}{2}h + \tfrac{1}{8}h^2 - \tfrac{1}{48}h^3, \qquad u_{1,3} = -h + \tfrac{1}{2}h^2 - \tfrac{1}{6}h^3,$$

$$u_{2,3} = -\tfrac{3}{2}h - \tfrac{1}{8}h^2 - \tfrac{1}{16}h^3.$$

When $s(0) = s_0 > 0$, the initial condition (4), boundary condition (3) and differential equation (12), give

$$u_{i,1} = g(x_i) \text{ and } \gamma_1^2 + b_1\gamma_1 - g''(s_0) = 0 \tag{28}$$

to determine γ_1, where $\gamma_j = \gamma(s_j - s_0)$ etc. Dividing the initial line $s = s_0$, $0 < x < s_0$ into m ($\geqslant 2$) intervals of length h allows the marching scheme (15)–(17) to start from $j = 1$ with $i = 2, 3, \ldots, m + j$. An expansion for γ near s_0 can be constructed, or the first time interval t_2 defined by

$$t_2 = \frac{h}{2}\left(\frac{1}{\gamma_1} + \frac{1}{\gamma_2}\right), \tag{29}$$

with $\gamma_2 = \gamma_1$ to start the iteration.

The essential feature of this method is that the boundary location in the (x, s) domain is known exactly, and the iteration is required to determine the boundary velocity and not boundary position through variable space or time steps. It is to be expected that similar accuracies in these different iteration schemes will result in better boundary position accuracy obtained by integration of a boundary velocity. The present iteration scheme is also very simple, and the example (7)–(10) with significant boundary acceleration shows that the first estimate of γ is adequate after an initial time interval. The present iteration arises through the non-linearity in $\hat{u}$ and γ which occurs for both linear and non-linear initial equations, so no further complexity is expected for an initial non-linear problem.

4. Numerical solutions

First consider the problem with boundary conditions (10) and exact solution (7)–(9). Figure 2 shows the exact boundary position over a range $0 \leqslant s \leqslant 5$, and fixed boundary value $-u(0, t)$ for $0 \leqslant t \leqslant 2.77$. Let T denote the computed value of t and γ the computed value of $\dot{s}$ at given position s, using $r = k = 0.5$, then Table 1 shows the comparison of exact and computed solutions, absolute errors and relative errors for $0 \leqslant s \leqslant 10$. The values shown for $0 \leqslant s \leqslant 2$ were computed with $h = 0.005$, those shown for $2 \leqslant s \leqslant 5$ with $h = 0.01$ for the *complete range* $0 \leqslant s \leqslant 5$ and those shown for $5 \leqslant s \leqslant 10$ with $h = 0.02$ for the *complete range* $0 \leqslant s \leqslant 10$. Thus, the pairs of values at $s = 2$ and $s = 5$ indicate the differences obtained with the different step lengths. The errors in boundary velocity are remarkably small, even with a coarse interval $h = 0.02$ over a range $0 \leqslant s \leqslant 10$ in which $\dot{s}$ increases from 0.5 to 147.9. The corresponding time errors, and hence boundary position at given time, are larger, but the relative error is still very small at $s = 10$.

Table 1. Exact solution comparison, boundary motion

s	t	T	$T-t$	$\frac{T-t}{t}$	$\dot{s}$	γ	$\gamma-\dot{s}$	$\frac{\gamma-\dot{s}}{\dot{s}}$
($h = 0.005$)								
0	0	0	0		0.5	0.5	0	
0.2	0.3636	0.3640	0.0004	0.0011	0.6052	0.6046	−0.0006	−0.0010
0.4	0.6664	0.6670	0.0007	0.0010	0.7214	0.7208	−0.0006	−0.0008
0.6	0.9218	0.9227	0.0008	0.0009	0.8499	0.8493	−0.0005	−0.0006
0.8	1.1398	1.1407	0.0010	0.0008	0.9918	0.9914	−0.0005	−0.0005
1.0	1.3272	1.3282	0.0010	0.0008	1.1487	1.1483	−0.0004	−0.0003
1.2	1.4895	1.4906	0.0011	0.0007	1.3221	1.3218	−0.0003	−0.0003
1.4	1.6310	1.6321	0.0011	0.0007	1.5138	1.5135	−0.0003	−0.0002
1.6	1.7548	1.7559	0.0011	0.0006	1.7255	1.7253	−0.0002	−0.0001
1.8	1.8636	1.8647	0.0012	0.0006	1.9596	1.9594	−0.0002	−0.0001
2.0	1.9595	1.9607	0.0011	0.0006	2.2183	2.2182	−0.0001	−0.0001
($h = 0.01$)								
2.0	1.9595	1.9618	0.0023	0.0012	2.2183	2.2180	−0.0003	−0.0001
2.3	2.0832	2.0855	0.0023	0.0011	2.6582	2.6580	−0.0002	−0.0001
2.6	2.1866	2.1889	0.0023	0.0010	3.1693	3.1692	−0.0001	−0.0000
2.9	2.2736	2.2759	0.0023	0.0010	3.7631	3.7631	−0.0000	−0.0000
3.2	2.3469	2.3492	0.0023	0.0010	4.4530	4.4530	−0.0000	−0.0000
3.5	2.4090	2.4113	0.0023	0.0010	5.2546	5.2546	0.0000	0.0000
3.8	2.4617	2.4640	0.0023	0.0009	6.1859	6.1859	0.0000	0.0000
4.1	2.5065	2.5088	0.0023	0.0009	7.2679	7.2679	0.0000	0.0000
4.4	2.5446	2.5469	0.0023	0.0009	8.5250	8.5250	0.0000	0.0000
4.7	2.5772	2.5794	0.0023	0.0009	9.9856	9.9856	0.0000	0.0000
5.0	2.6050	2.6072	0.0023	0.0009	11.6825	11.6825	0.0000	0.0000
($h = 0.02$)								
5.0	2.6050	2.6094	0.0044	0.0017	11.6825	11.6827	0.0002	0.0000
6.0	2.6718	2.6762	0.0044	0.0017	19.5855	19.5858	0.0003	0.0000
7.0	2.7117	2.7161	0.0044	0.0016	32.6154	32.6159	0.0005	0.0000
8.0	2.7358	2.7402	0.0044	0.0016	54.0982	54.0989	0.0007	0.0000
9.0	2.7503	2.7547	0.0044	0.0016	89.5171	89.5181	0.0010	0.0000
10.0	2.7591	2.7635	0.0044	0.0016	147.913	147.914	0.0012	0.0000

Denote the exact boundary value $u(0, t)$ by u_t, the exact boundary value $u(0, T)$ by u_T and the computed boundary value $\hat{u}(0, s)$ by U. Table 2 compares U with u_t and with u_T, both comparisons showing small relative errors over the entire range, with $|(U-u_T)/u_T|$ the least. That is, U is better interpreted as the value at the calculated time rather than at the exact time when the moving boundary is at s, which is the more useful comparison. No iterations were required with $h = 0.005$, and only one iteration for the first 17 steps with $h = 0.01$. Also no iterations were required with $h = 0.02$, but recall that the halt criterion adopted used a change proportional to h^2, to be consistent with the finite difference approximation, so increasing h decreases the demanded accuracy.

Comparison computations were made with the weightings $r = k = 1$, giving T differences of magnitude 10^{-4}, γ differences of magnitude 2×10^{-4}, and U differences of magnitude 10^{-4}, with $h = 0.01$, all much smaller than the errors.

The Gupta and Kumar [8] problem (5) was computed with $r = k = 0.5$, and both $h = 0.01$, their smallest step length, for $0 \leqslant s \leqslant 5$, and $h = 0.02$ for $0 \leqslant s \leqslant 10$. Table 3 shows the

Table 2. Exact solution comparison, fixed boundary values

s	$-u_t$	$-U$	$-U+u_t$	$\frac{U-u_t}{u_t}$	$-u_T$	$-U+u_T$	$\frac{U-u_T}{u_T}$
($h = 0.005$)							
0	0	0	0		0	0	
0.2	0.0952	0.0953	0.0001	0.0013	0.0953	0.0000	0.0002
0.4	0.1813	0.1815	0.0002	0.0013	0.1815	0.0000	0.0003
0.6	0.2592	0.2595	0.0004	0.0014	0.2594	0.0001	0.0003
0.8	0.3297	0.3301	0.0005	0.0014	0.3300	0.0001	0.0004
1.0	0.3935	0.3940	0.0005	0.0014	0.3938	0.0002	0.0004
1.2	0.4512	0.4518	0.0006	0.0014	0.4516	0.0002	0.0005
1.4	0.5034	0.5041	0.0007	0.0014	0.5038	0.0003	0.0005
1.6	0.5507	0.5514	0.0007	0.0013	0.5511	0.0003	0.0005
1.8	0.5934	0.5942	0.0008	0.0013	0.5939	0.0003	0.0006
2.0	0.6321	0.6330	0.0008	0.0013	0.6326	0.0004	0.0006
($h = 0.01$)							
2.0	0.6321	0.6338	0.0017	0.0027	0.6330	0.0008	0.0012
2.3	0.6834	0.6852	0.0018	0.0027	0.6843	0.0009	0.0013
2.6	0.7275	0.7294	0.0019	0.0026	0.7285	0.0009	0.0013
2.9	0.7654	0.7674	0.0020	0.0026	0.7664	0.0010	0.0013
3.2	0.7981	0.8002	0.0021	0.0026	0.7991	0.0011	0.0013
3.5	0.8262	0.8284	0.0022	0.0026	0.8273	0.0011	0.0013
3.8	0.8504	0.8526	0.0022	0.0026	0.8515	0.0011	0.0013
4.1	0.8713	0.8735	0.0023	0.0026	0.8723	0.0012	0.0014
4.4	0.8892	0.8915	0.0023	0.0026	0.8903	0.0012	0.0014
4.7	0.9046	0.9070	0.0023	0.0026	0.9057	0.0012	0.0014
5.0	0.9179	0.9203	0.0024	0.0026	0.9190	0.0013	0.0014
($h = 0.02$)							
5.0	0.9179	0.9227	0.0048	0.0052	0.9200	0.0027	0.0029
6.0	0.9502	0.9552	0.0050	0.0052	0.9542	0.0028	0.0030
7.0	0.9698	0.9749	0.0051	0.0052	0.9720	0.0029	0.0030
8.0	0.9817	0.9868	0.0051	0.0052	0.9839	0.0029	0.0029
9.0	0.9889	0.9940	0.0051	0.0052	0.9911	0.0029	0.0030
10.0	0.9926	0.9977	0.0052	0.0052	0.9948	0.0030	0.0030

computed γ, time T_M, and boundary value $\hat{u}(0, s) = U_M$, with the Gupta-Kumar values T_G, U_G, and integral method approximation values T_I, U_I for comparison. The present method shows no indication of instability at the maximum range, $s = 10$, taken, but Gupta and Kumar stopped their computation at $s = 3$, so no subsequent comparison is possible. It is clear that the Gupta-Kumar solution is much closer to the present solution than to the approximate integral method solution they presented to confirm the accuracy of their computation, suggesting better accuracy than claimed. They set a relative error of 0.005 to halt the iteration but give no information about the number of iterations requried or computing time in general.

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Table 3. Comparison with Gupta-Kumar solution

s	γ	T_M	T_G	T_I	U_M	U_G	U_I
($h = 0.01$)							
0	1.0	0	0	0	0	0	0
0.2	0.8555	0.2173	0.2172	0.2181	0.1859	0.1858	0.1854
0.4	0.7625	0.4657	0.4656	0.4675	0.3527	0.3526	0.3531
0.6	0.6956	0.7408	0.7406	0.7444	0.5082	0.5082	0.5110
0.8	0.6440	1.0400	1.0395	1.0568	0.6558	0.6560	0.6623
1.0	0.6023	1.3614	1.3604	1.3727	0.7975	0.7981	0.8090
1.2	0.5677	1.7037	1.7015	1.7217	0.9344	0.9353	0.9521
1.4	0.5381	2.0657	2.0621	2.0928	1.0673	1.0689	1.0923
1.6	0.5126	2.4467	2.4413	2.4854	1.1970	1.1994	1.2301
1.8	0.4901	2.8459	2.8382	2.8991	1.3239	1.3272	1.3659
2.0	0.4702	3.2626	3.2522	3.3333	1.4482	1.4524	1.5000
2.2	0.4524	3.6964	3.6829	3.7878	1.5704	1.5758	1.6326
2.4	0.4363	4.1467	4.1295	4.2623	1.6905	1.6976	1.7639
2.6	0.4216	4.6131	4.5916	4.7564	1.8089	1.8183	1.8941
2.8	0.4082	5.0953	5.0684	5.2700	1.9257	1.9378	2.0232
3.0	0.3959	5.5930	5.5599	5.8028	2.0410	2.0552	2.1514
3.2	0.3845	6.1057			2.1550		
3.4	0.3739	6.6332			2.2676		
3.6	0.3641	7.1753			2.3792		
3.8	0.3549	7.7317			2.4896		
4.0	0.3463	8.3022			2.5990		
5.0	0.3103	11.3592			3.1327		
($h = 0.02$)							
5.0	0.3104	11.3484			3.1351		
6.0	0.2826	14.7298			3.6507		
7.0	0.2604	18.4205			4.1517		
8.0	0.2421	22.4073			4.6407		
9.0	0.2267	26.6796			5.1198		
10.0	0.2135	31.2281			5.5901		

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